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Degree of L_p-Approximation by Integral Schoenberg Splines

MANFRED W. MÜLLER

Universität Dortmund, D4600 Dortmund 50, West Germany

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1. INTRODUCTION

Let Δ_n: $0 = x_0 < x_1 < \cdots < x_n = 1$ ($n \in \mathbb{N}$) be a finite partition of the interval $I = [0, 1]$. This partition is extended to a sequence $\Delta_{n,k} := \{x_i\}_{i=-k}^{n+k}$ of so-called *knots* by setting $x_{-k} = \cdots = x_{-1} = 0$ and $x_{n+1} = \cdots = x_{n+k} = 1$ ($k \in \mathbb{N}$).

Schoenberg [12] has constructed a generalization of the Bernstein polynomials, by associating with a function $f: I \to \mathbb{R}$ the spline function of degree k (order $k + 1$)

$$S_{n,k}f(x) = \sum_{j=-k}^{n-1} f(\xi_{j,k})\, N_{j,k}(x), \qquad 0 \leqslant x \leqslant 1, \tag{1.1}$$

with knots from Δ_n. $S_{n,k}f$ is to be regarded as an approximation to f on I. The function f to be approximated is evaluated at discrete *nodes* $\xi_{j,k}$ depending on k and $\Delta_{n,k}$:

$$\xi_{j,k} = \frac{x_{j+1} + x_{j+2} + \cdots + x_{j+k}}{k} \qquad (-k \leqslant j \leqslant n-1). \tag{1.2}$$

These nodes satisfy:

$$\begin{gathered} 0 = \xi_{-k,k} < \xi_{-k+1,k} < \cdots < \xi_{n-1,k} = 1, \\ \xi_{j+1,k} - \xi_{j,k} = \frac{x_{j+k+1} - x_{j+1}}{k}. \end{gathered} \tag{1.3}$$

The weights $N_{j,k}(x)$ in (1.1) are known as *normalized B*-splines:

$$N_{j,k}(x) = \frac{x_{j+k+1} - x_j}{k+1} M_{j,k}(x), \tag{1.4}$$

where the B-spline $M_{j,k}(x)$ (see [2]) is a spline of degree k, the $(k+1)$th divided difference of (the function of t; x fixed)

$$\begin{aligned} M(x; t) = (k+1)(t-x)_+^k &= (k+1)(t-x)^k, && t \geqslant x, \\ &= 0, && t < x, \end{aligned}$$

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on $x_j, \dots, x_{j+k+1}$. Thus

$$M_{j,k}(x) := M(x; x_j, x_{j+1}, \dots, x_{j+k+1}).$$

For later reference we list some known facts about B-splines (see [2, 3, 9]):

$$N_{j,k}(x) \geq 0, \qquad \operatorname{supp} N_{j,k}(x) = [x_j, x_{j+k+1}]; \tag{1.5}$$

$$\sum_{j=-k}^{n-1} N_{j,k}(x) = 1, \qquad \sum_{j=-k}^{n-1} \xi_{j,k} N_{j,k}(x) = x, \qquad \int_0^1 M_{j,k}(x)\, dx = 1. \tag{1.6}$$

The Schoenberg spline operators $S_{n,k}$ are linear and positive, reproduce linear functions, and are variation diminishing (see [13]). Marsden [9] proved that they are even a linear approximation method on the space $C(I)$ (with $\|\cdot\|_\infty$ the usual sup-norm on I) in the following sense: $\|S_{n,k}f - f\|_\infty \to 0$ for $f \in C(I)$ as the mesh of Δ_n, $|\Delta_n| := \max_i (x_{i+1} - x_i)$, goes to zero. (For $k = 1$ this method reduces to linear interpolation.)

In Section 2, we shall extend Schoenberg's approximation method to a method for the L_p-approximation of functions $f \in L_p(I)$, $1 \leq p \leq \infty$, the space of real-valued pth power integrable functions on I, with $\|\cdot\|_p$ the usual L_p-norm on I. The corresponding spline operators $T_{n,k}$ will roughly speaking be constructed replacing $f(\xi_{j,k})$ in (1.1) by an integral mean of f over a suitable small interval around the node $\xi_{j,k}$. Therefore we shall refer to them as "*integral Schoenberg spline operators*."

In Section 3, the L_p-norm of the difference between a function $f \in L_p(I)$ and the associated integral Schoenberg spline $T_{n,k}f$ is estimated in terms of the first-order integral modulus of continuity $\omega_{1,p}(f, \cdot)$. The main result will be that $\|f - T_{n,k}f\|_p = O(\omega_{1,p}(f, |\Delta_n|))$.

It should be observed that the right-hand side of this estimation is of order $O(n^{-\alpha})$ if the partition Δ_n is equidistant and if f is belonging to a Lipschitz class $\mathrm{Lip}(\alpha, L_p)$. The method of proof is smoothing, i.e., f is approximated first by a function g with g' in $L_p(I)$ and then g is approximated by $T_{n,k}\, g$ (see, e.g., [4]). The connection between these two processes is given via the K-functional of Peetre [10].

2. L_p-Approximation

Applying the first-derivative operator D to (1.1), we obtain easily, utilizing a lemma of Marsden [9, p. 32],

$$DS_{n,k+1}f(x) = \sum_{j=-k}^{n-1} \frac{f(\xi_{j,k+1}) - f(\xi_{j-1,k+1})}{\xi_{j,k+1} - \xi_{j-1,k+1}} N_{j,k}(x), \qquad 0 \leq x \leq 1, \tag{2.1}$$

where $\xi_{i,k+1}$ $(-k-1 \leqslant i \leqslant n-1)$ are nodes given by (1.2) with k replaced by $k+1$ and the functions $N_{j,k}(x)$ are the normalized B-splines given by (1.4). $DS_{n,k+1}f$ is a spline function of degree k.

If $f \in L_p(I)$, consider the indefinite integral $F(x) = \int_0^x f(t)\,dt$. Equation (2.1) applied to the (absolutely continuous) function F gives for $0 \leqslant x \leqslant 1$

$$T_{n,k}f(x) := DS_{n,k+1}F(x) = \sum_{j=-k}^{n-1} \frac{N_{j,k}(x)}{\xi_{j,k+1} - \xi_{j-1,k+1}} \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} f(t)\,dt. \tag{2.2}$$

$T_{n,k}f$ is again a spline function of degree k. For reasons mentioned in Section 1 we shall denote it as integral Schoenberg spline of degree k. Its representation (2.2) can be simplified observing (1.4) together with

$$\xi_{j,k+1} - \xi_{j-1,k+1} = \frac{x_{j+k+1} - x_j}{k+1} \tag{2.3}$$

to

$$T_{n,k}f(x) = \sum_{j=-k}^{n-1} M_{j,k}(x) \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} f(t)\,dt, \qquad 0 \leqslant x \leqslant 1. \tag{2.4}$$

The operators $T_{n,k}$ are linear positive and preserve the identity. In a certain sense they can serve as a linear approximation method on the space $L_p(I)$, which is shown by the following theorem.

THEOREM 1. *For $f \in L_p(I)$, $1 \leqslant p \leqslant \infty$, there holds*

(i) $\lim_{|\Delta_n| \to 0} \| T_{n,k}f - f \|_p = 0 \qquad (k \in \mathbb{N}$ *fixed*$)$,

(ii) $\lim_{k \to \infty} \| T_{n,k}f - f \|_p = 0 \qquad (n \in \mathbb{N}$ *fixed*$)$.

Proof. The spline (2.4) can be considered as a singular integral of the type

$$T_{n,k}f(x) = \int_0^1 H_{n,k}(x, t)\, f(t)\,dt$$

with the positive kernel

$$H_{n,k}(x, t) = \sum_{j=-k}^{n-1} M_{j,k}(x)\, 1_{j-1,k+1}(t),$$

where $1_{j-1,k+1}$ is the characteristic function of the interval $[\xi_{j-1,k+1}, \xi_{j,k+1}]$ with respect to I. Utilizing (1.4), (1.6), and (2.3) we have for all n and k and all x or t, respectively,

$$\int_0^1 H_{n,k}(x, t)\,dt = \sum_{j=-k}^{n-1} M_{j,k}(x)(\xi_{j,k+1} - \xi_{j-1,k+1}) = \sum_{j=-k}^{n-1} N_{j,k}(x) = 1, \tag{2.5}$$

$$\int_0^1 H_{n,k}(x, t)\,dx = \sum_{j=-k}^{n-1} 1_{j-1,k+1}(t) \int_0^1 M_{j,k}(x)\,dx = 1. \tag{2.6}$$

Step one. The sequence of operators $(T_{n,k})$ is uniformly bounded in n resp. k.

First, assume $p > 1$. Using Hölder's inequality with $p^{-1} + q^{-1} = 1$, we obtain for an arbitrary $f \in L_p(I)$ by (2.5)

$$|T_{n,k}f(x)| \leq \left\{\int_0^1 H_{n,k}(x, t) \, |f(t)|^p \, dt\right\}^{1/p},$$

from there by (2.6) and Fubini's theorem

$$\|T_{n,k}f\|_p \leq \left\{\int_0^1 \int_0^1 H_{n,k}(x, t) \, |f(t)|^p \, dx \, dt\right\}^{1/p}$$

$$= \left\{\int_0^1 |f(t)|^p \left[\int_0^1 H_{n,k}(x, t) \, dx\right] dt\right\}^{1/p} = \|f\|_p$$

and hence $\|T_{n,k}\|_p \leq 1$ for all n and $k \in \mathbb{N}$. The case $p = 1$ may be dealt with in the same manner; the proof is simpler and does not require (2.5).

Step two. (i) resp. (ii) holds for the dense subspace $C(I)$ of $L_p(I)$.

Using (1.1), (2.4), and the very useful relation

$$\sum_{j=-k}^{n-1} M_{j,k}(x) \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} dt = 1, \tag{2.7}$$

we have for an arbitrary $x \in I$

$$|T_{n,k}f(x) - S_{n,k}f(x)| \leq \sum_{j=-k}^{n-1} M_{j,k}(x) \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} |f(t) - f(\xi_{j,k})| \, dt. \tag{2.8}$$

In view of

$$\begin{aligned} \xi_{j,k+1} - \xi_{j-1,k+1} &\leq \min\left(\|\Delta_n\|, \frac{1}{k+1}\right), \\ \xi_{j,k+1} - \xi_{j,k} &\leq \min\left(\frac{\|\Delta_n\|}{2}, \frac{1}{k+1}\right) \end{aligned} \tag{2.9}$$

we obtain from (2.8) and (2.7)

$$\begin{aligned} \|T_{n,k}f - S_{n,k}f\|_p &\leq \|T_{n,k}f - S_{n,k}f\|_\infty \\ &\leq \omega_\infty(f, \|\Delta_n\|) \qquad (k \text{ fixed, } n \text{ sufficiently large}), \\ &\leq \omega_\infty\left(f, \frac{1}{k+1}\right) \qquad (n \text{ fixed, } k \text{ sufficiently large}). \end{aligned} \tag{2.10}$$

where $\|\cdot\|_\infty$ denotes here the sup-norm on I and $\omega_\infty(f, \cdot)$ is the ordinary modulus of continuity with respect to this norm. Now

$$\|f - T_{n,k}\|_p \leqslant \|f - S_{n,k}f\|_p + \|S_{n,k}f - T_{n,k}f\|_p .$$

For $|\Delta_n| \to 0$ (resp. $k \to \infty$) each term goes to zero, the first one on account of a result of Marsden [9, Theorem 3], the second one by (2.9), which proves the assertion of step two.

The rest of the proof follows by the density of $C(I)$ in $L_p(I)$ with respect to the L_p-norm since the norms of the operators $T_{n,k}$ are in any of the two cases bounded by 1.

Remarks. (1) If $n = 1$, then the integral Schoenberg splines reduce to the Kantorovič polynomials [7] (which are obtained from the Bernstein polynomials in the same way as our splines $T_{n,k}f$ from the Schoenberg splines $S_{n,k}f$) and Theorem 1, (ii) turns to a well-known result of Lorentz [8, Theorem 2.1.2].

(2) For a certain modification of the operators $T_{n,k}$ Scherer [11] proved an approximation theorem of the same kind as part (i) of Theorem 1.

As an application of Theorem 1 we obtain the following criterion of compactness for a bounded subset

$$K := \{f \in L_p(I) \mid \|f\|_p \leqq M, M \text{ a positive constant}\}$$

of $L_p(I)$: *K is compact with respect to the L_p-norm iff $\|f - T_{n,k}f\|_p \to 0$ ($|\Delta_n| \to 0$) uniformly for all $f \in K$.*

The method of proof is quite similar to an argument given by Lorentz [8, p. 33] for Kantorovič polynomials using the fact that by Hausdorff's criterion of compactness in complete metric spaces (see [5, p. 108]) K is compact iff for each $\epsilon > 0$ there is a finite ϵ-net.

3. Degree of L_p-Approximation

In this section only splines of fixed degree k will be considered. Let $L_p^1(I)$ be the space of those functions $f \in L_p(I)$, with f absolutely continuous, $f' \in L_p(I)$ and the norm $\|f\|_p^1 = \|f\|_p + \|f'\|_p$.

The following theorem gives an upper bound for the degree of L_p-approximation of integral Schoenberg splines to "smooth" functions $f \in L_p^1(I)$. It will be the key for proving our main result. For its proof we need the following

LEMMA.

$$\sum_{j=-k}^{n-1} M_{j,k}(x) \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} |t-x|\, dt \leq (k+1)\,|\Delta_n| \qquad (x \in I).$$

Proof. Fix $x \in \operatorname{supp} M_{j,k}(x)$. In view of $\operatorname{supp} M_{j,k}(x) = [x_j, x_{j+k+1}]$ and $\xi_{j-1,k+1}, \xi_{j,k+1} \in \operatorname{supp} M_{j,k}(x)$ we have $|t-x| \leq (k+1)\,|\Delta_n|$ and thus by (2.7)

$$\sum_{j=-k}^{n-1} M_{j,k}(x) \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} |t-x|\, dt \leq (k+1)\,|\Delta_n| \sum_{j=-k}^{n-1} M_{j,k}(x) \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} dt$$
$$= (k+1)\,|\Delta_n|.$$

THEOREM 2. *For $f \in L_p^1(I)$, $1 \leq p < \infty$, there holds*

$$\| T_{n,k}f - f \| \leq (k+1)\, \| f' \|_p\, |\Delta_n|.$$

Proof. Fix $x \in I$. Then by (2.7)

$$|T_{n,k}f(x) - f(x)| = \left| \sum_{j=-k}^{n-1} \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} M_{j,k}(x) \left(\int_x^t f'(u)\, du \right) dt \right|$$
$$\leq \sum_{j=-k}^{n-1} \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} \int_x^t \{ M_{j,k}(x)\, |f'(u)|\, du \}\, dt.$$

Applying twice Hölder's inequality with $p^{-1} + q^{-1} = 1$, then Cauchy–Schwarz's inequality, and the lemma yields

$$|T_{n,k}f(x) - f(x)|$$
$$\leq \sum_{j=-k}^{n-1} M_{j,k}(x) \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} \left\{ |t-x|^{1/q} \left(\int_x^t |f'(u)|^p\, du \right)^{1/p} \right\} dt$$
$$\leq \sum_{j=-k}^{n-1} M_{j,k}(x) \left(\int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} |t-x|\, dt \right)^{1/q} \left(\int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} \int_x^t |f'(u)|^p\, du\, dt \right)^{1/p}$$
$$\leq \left\{ \sum_{j=-k}^{n-1} M_{j,k}(x) \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} |t-x|\, dt \right\}^{1/q}$$
$$\times \left\{ \sum_{j=-k}^{n-1} M_{j,k}(x) \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} \int_x^t |f'(u)|^p\, du\, dt \right\}^{1/p}$$
$$\leq (k+1)^{1/q}\, |\Delta_n|^{1/q} \left\{ \sum_{j=-k}^{n-1} M_{j,k}(x) \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} \int_x^t |f'(u)|^p\, du\, dt \right\}^{1/p}.$$

From this follows in view of supp $M_{j,k}(x) = [x_j, x_{j+k+1}]$, (1.6), (2.9), and $x_{j+k+1} - \xi_{j,k+1} > 0$

$$\| T_{n,k}f - f \|_p$$

$$\leqq (k+q)^{1/q} \mid \Delta_n \mid^{1/q} \left\{ \sum_{j=-k}^{n-1} \int_0^1 \left[M_{j,k}(x) \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} \int_x^{x_{j+k+1}} \mid f'(u) \mid^p du\, dt \right] dx \right\}^{1/p}$$

$$\leqq (k+1)^{1/q} \mid \Delta_n \mid^{1/q} \left\{ \sum_{j=-k}^{n-1} \mid \Delta_n \mid \int_{x_j}^{x_{j+k+1}} \mid f'(u) \mid^p du \right\}^{1/p}$$

$$\leqq (k+1)^{1/q} \mid \Delta_n \mid \left\{ (k+1) \int_0^1 \mid f'(u) \mid^p du \right\}^{1/p}$$

$$\leqq (k+1) \| f' \|_p \mid \Delta_n \mid .$$

This completes the proof.

Remark. For the approximation of $f \in L_p^1(I)$ by Schoenberg splines Scherer [11, Satz 2] obtained a result of the same kind.

In what follows we will measure smoothness using the K-functional of Peetre [10]. It is for $f \in L_p(I)$, $1 \leqslant p \leqslant \infty$, defined by

$$K_p(t, f) = \inf_{g \in L_p^1} (\| f - g \|_p + t \| g' \|_p) \qquad (0 \leqslant t \leqslant 1). \tag{3.1}$$

Roughly speaking the K-functional is a seminorm on $L_p(I)$ measuring the degree of approximation of a function $f \in L_p(I)$ by smoother functions $g \in L_p^1(I)$ with simultaneous control on the size of $\| g' \|_p$ (see [4]).

The more classical measure for smoothness, the integral modulus of continuity, which for $f \in L_p(I)$, $1 \leqslant p \leqslant \infty$, is defined by

$$\omega_{1,p}(f, h) := \sup_{0 < t \leqq h} \| f(\cdot + t) - f(\cdot) \|_p \, (I_t) \tag{3.2}$$

(where $\| \cdot \| (I_t)$ is indicating that the L_p-norm is to be taken over the interval $I_t = [0, 1 - t]$) is in a certain sense equivalent to the K-functional. Johnen [6, Proposition 6.1] proved that there are constants $c_1 > 0$ and $c_2 > 0$, independent of f and p, such that

$$c_1 \omega_{1,p}(f, t) \leqslant K_p(t, f) \leqslant c_2 \omega_{1,p}(f, t) \qquad (0 \leqslant t \leqslant 1). \tag{3.3}$$

THEOREM 3. *For $f \in L_p(I)$, $1 \leqslant p \leqslant \infty$, there holds*

$$\| T_{n,k}f - f \|_p \leqslant M \omega_{1,p}(f, \mid \Delta_n \mid),$$

where M is some positive constant, independent of f and p.

Proof. In view of Theorem 2 and $\| T_{n,k} \|_p \leqslant 1$, we have

$$\begin{aligned} \| T_{n,k}h - h \|_p &\leqslant 2 \| h \|_p, & h \in L_p(I), \\ &\leqslant (k+1) \| h' \|_p \, |\Delta_n|, & h \in L_p^1(I). \end{aligned}$$

When $f \in L_p(I)$ and g is an arbitrary function from $L_p^1(I)$, then

$$\begin{aligned} \| T_{n,k}f - f \|_p &\leqslant \| T_{n,k}(f-g) - (f-g) \|_p + \| T_{n,k}g - g \|_p \\ &\leqslant 2 \| f - g \|_p + (k+1) \| g' \|_p \, |\Delta_n| \\ &\leqslant 2(\| f - g \|_p + k \, |\Delta_n| \, \| g' \|_p). \end{aligned}$$

Taking now the infimum over all $g \in L_p^1(I)$ on the right-hand side, using the definition of the K-functional and observing (3.3), we find

$$\| T_{n,k}f - f \|_p \leqslant 2K(k \, |\Delta_n|, f) \leqslant 2c_1 \omega_{1,p}(f, k \, |\Delta_n|).$$

Since $\omega_{1,p}(f, k \, |\Delta_n|) \leqslant k \omega_{1,p}(f, |\Delta_n|)$ for $k \in \mathbb{N}$ the theorem is proved.

COROLLARY. *If* $f \in \mathrm{Lip}(\alpha, L_p)$ $(0 < \alpha \leqslant 1)$, *then*

$$\| T_{n,k}f - f \|_p = O(|\Delta_n|^{\alpha}).$$

Here the Lipschitz class $\mathrm{Lip}(\alpha, L_p)$ of order α with respect to the L_p-norm is defined as the collection of all functions $f \in L_p(I)$ with the property $\omega_{1,p}(f, t) = O(t^{\alpha})$ $(t \to 0+)$. To make these last results still more transparent, we are considering the family of spaces $[L_p^1, L_p]_\alpha$, $0 \leqslant \alpha \leqslant 1$, consisting of all functions $f \in L_p(I)$, for which

$$\| f \|_{[L_p^1, L_p]_\alpha} := \sup_{0 < t \leqslant 1} t^{-\alpha} K_p'(t, f) < \infty, \tag{3.4}$$

where $K_p'(t, \cdot)$ is a modified K-functional on $L_p(I)$ given by

$$\begin{aligned} K_p'(t, f) &:= \inf_{g \in L_p^1} (\| f - g \|_p + t \| g \|_p^1) \\ &= \inf_{g \in L_p^1} (\| f - g \|_p + t(\| g \|_p + \| g' \|_p)) \qquad (0 < t \leqslant 1) \end{aligned}$$

and connected to $K_p(t, \cdot)$ by

$$K_p(t, f) \leqslant K_p'(t, f) \leqslant t \| f \|_p + 2K_p(t, f) \tag{3.5}$$

(see [6, p. 300]). The spaces $[L_p^1, L_p]_\alpha$ are complete under the norm (3.4) and intermediate between $L_p^1(I)$ and $L_p(I)$, i.e., continuously embedded between these two spaces (see, e.g., [1, p. 168]).

Utilizing (3.3) and (3.5) it can easily be proved that $f \in [L_p{}^1, L_p]_\alpha$ is equivalent to $f \in \mathrm{Lip}(\alpha, L_p)$, $0 < \alpha \leqslant 1$. Thus the corollary tells that especially for equidistant partitions of the interval I (i.e., $|\Delta_n| = n^{-1}$) the elements of an intermediate space $[L_p{}^1, L_p]_\alpha$ between $L_p{}^1(I)$ and $L_p(I)$ are approximated by our method with respect to the L_p-norm of the order $O(n^{-\alpha})$ if n goes to infinity. The case of fixed knots and degree k tending to infinity will be treated in a forthcoming paper.

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